Exercise 34

(a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

Solution

The aim is to evaluate y' at $x = 1$ and $y = -2$ in order to find the slope there. Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 + 3x^2)$$

$$(2y) \cdot \frac{d}{dx}(y) = \frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2)$$

$$(2y) \cdot y' = (3x^2) + 3(2x)$$

Solve for y'.

$$y' = \frac{3x^2 + 6x}{2y}$$

Evaluate y' at $x = 1$ and $y = -2$.

$$y'(1, -2) = \frac{3(1)^2 + 6(1)}{2(-2)} = -\frac{9}{4}$$

Therefore, the equation of the tangent line to the curve represented by $y^2 = x^3 + 3x^2$ at $(1, -2)$ is

$$y + 2 = -\frac{9}{4}(x - 1).$$

The curve has horizontal tangents where $y' = 0$, that is,

$$y' = \frac{3x^2 + 6x}{2y} = \frac{3x^2 + 6x}{\pm 2\sqrt{x^3 + 3x^2}} = \frac{3x^2 + 6x}{\pm 2x\sqrt{x + 3}} = \frac{3x + 6}{\pm 2\sqrt{x + 3}} = 0 \quad \rightarrow \quad 3x + 6 = 0 \quad \rightarrow \quad x = -2.$$

Plug this value of x into the curve's equation to get the corresponding value of y.

$$x = -2: \quad y^2 = (-2)^3 + 3(-2)^2 = 4 \quad \rightarrow \quad y = \{-2, 2\}$$

Therefore, the curve has horizontal tangents at $(-2, -2)$ and $(-2, 2)$.

Below is a graph of the curve and the tangent line at $(1, -2)$ and the horizontal tangents.